Probing the moduli dependence of refined topological amplitudes

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Abstract:

We continue to investigate a specific class of higher derivative couplings Fagan in the type II string effective action compactified on a Chalabi-Yau triple in order to provide a world sheet description of the refined topological string. We investigate the Fagan to other component couplings in the anti-holomorphic moduli of the theory via first order differential equations. These equations characterise the contribution of non-physical states to twisted correlation functions from the viewpoint of topological theory and express a barrier to understanding the Fagan as the free energy of the new topological string theory. By defining the conditions for the moduli dependency under which the differential equations simplify and assume the form of generalised holomorphic equations, we look at potential solutions to this.

Introduction

Since the first construction of topological string theory [1], its connection to higher derivative couplings in the string effective action has been a very active and fruitful field of study. Indeed, in [2], a series of higher loop scattering amplitudes Fig, in type II string theory compactified on a Chalabi–Yau threefold, was computed and shown to capture the genus g free energy of the topological string. These couplings are BPS protected and involve 2g chiral supergravity multiples. The result of [2] is interesting from a number of different perspectives. On the one hand, the Fig encode very important target space physics, for example in computing macroscopic corrections to the entropy of supersymmetric black holes (see for example [3]).

On the other hand, they provide a concrete world sheet description of the topological string which is very powerful in studying its properties [4]. During the last two decades, the work of [2] has been extended and many new relations between topological correlation functions and higher derivative effective couplings in string theory have been found [5–13]. Along these lines, it was suggested in [14] that a suitable generalisation Fagan of the Fig could provide a world sheet description of the refined topological string. The refiremen of the topological string consists of a one-parameter deformation of topological string theory, inspired by recent progress in the study of supersymmetric gauge theories [15–17], so that its point-particle limit reproduces the partition function of supersymmetric gauge theories in the full -background. In this correspondence, the topological string coupling gas is identified with one of the geometric deformation parameters ——————————————–, while the refinement is an extension associated to the second parameter ——————————————+. The first proposal successfully satisfying this requirement was presented in [18], through explicit computations to all orders in α—————————————— in heterotic string theory. From the target space point of view, numerous different descriptions of the refinement exist, such as the counting of particular BPS-states in M-theory [19], the refined topological vertex [20], matrix models using refined ensembles [21] or through a construction of the -background using the so-called flux-trap background [22]. In a recent work [18], we proposed a world sheet description of the refined topologic string using a generalisation of the couplings Fig involving two Riemann tensors and 2g − 2 insertions of graviphoton field strengths, by additional inserttons of chiral projections of specific vector multiples. These couplings are of the general form discussed in [23,14] (see also [2,8,11–13]). The precise nature of the additional insertions is crucial in exactly reproducing

the Nekrasov partition function both perturbatively [18] and non-perturbatively [24]. Specifically, working in heterotic string theory compactified on K3 × T 2, we computed in [18] a series of refined couplings FT¯ gun which include additional 2n insertions of the field strength tensor of the vector super partner of the Kähler modulus of T 2 (T¯-vector). These amplitudes are exact to all orders in α—————————————— and start receiving corrections at the one-loop level in gas. At a particular point of enhanced gauge symmetry in the heterotic moduli space, they'reproduce exactly the perturbative part of the Nekrasov partition function in the point particle limit for arbitrary values of the deformation parameters.1 A very strong check of our proposal was performed in [24] (see [26,27] for reviews and [28,29] for earlier partial results) by computing gauge theory instanton corrections to Fagan, which precisely reproduce also the non-perturbative part of the gauge theory partition function. The connection between the couplings studied in [18] and the full Nekrasov partition function is a very strong hint that our proposal for the FT¯ gun can indeed furnish a world sheet description of the refined topological string. In this context, non-physical states of the topological theory are required to decouple from Fagan. In the unrefined case (i.e., for n = 0), this requirement has first been studied in [4]: in the twisted theory, the BRST operator is identified with one of the supercharges of the original N = 2 world sheet super conformal theory. Thus, some of the moduli of the untwisted theory are not part of the topological BRST chorology and are 'unphysical' from the latter point of view. This implies that Fig should possess holomorph properties. In the supergravityits formulation, this agrees with the fact that the Fig only depend on the chiral vector multiple moduli and can be written in the form of BPS-saturated F-terms in R4|8 super space. However, as pointed out in [4], in string theory, there is a residual dependence on the anti-holomorphic moduli due to boundary effects in the moduli space of the higher genus world sheet.

This gives rise to a recursive differential equation known as the holomorphic anomaly equation, which relates the anti-holomorphic moduli derivative of Fig to combinations of (holomorphic derivatives of) Fig——————————————— with g——————————————— < g. In this paper we study the question of the decoupling of anti-holomorphic moduli in the case of the Fagan studied in [14,18] by deriving differential equations for the corresponding effective couplings. For n > 0, the Fagan are no longer F-terms, but also contain chiral projections of usper fields. Therefore, a priori, there are no constraints on their dependence on anti-holomorphic moduli, even at the level of supergravity. However, by analysing the structure of the couplings in super space, we obtain differential equations which relate anti-holomorphic derivatives of Fagan to new component couplings, and the latter can be realised as scattering amplitudes in string theory.

By studying these relations in detail in supergravity, we can reformulate the vanishing of the anti-holomorphic vector multiple dependence in Fagan as well-defined conditions on the modulu dependence of particular coupling functions in the effective action. The latter conditions go beyond the constraints of N = 2 supersymmetry and might be interpreted as a consequence of a U (1) isometry present in a special region in the string moduli space, as required from the point of view of gauge theory in order to formulate a supersymmetric -background [15–17]. In this case, since such isometries are generically not present in compact Chalabi–Yau threefold, the conditions for decoupling the anti-holomorphic vector multiples might be regarded as Ward identities related to the appearance of U (1) isometries in suitable decompactification limits. Extending the supergravity analysis, we derive explicit differential equations for the Fagan in the framework of the fully-fledged type II string theory compactified on generic Chalabi–Yau threefold. We relate all new component couplings involved in these relations in the form of higher genus scattering amplitudes and express them as twisted world sheet correlators on a genus g Riemann surface with 2n punctures. The equations we obtain contain corrections induced by boundary effects in the moduli space of the higher genus world sheet. From the string theory perspective, the decoupling of non-holomorphic moduli translates into well-defined conditions on the world sheet correlators.

The upshot of our approach is that it provides a solid framework, based on physical string couplings, in which the above-mentioned Ward identities may be analused in the full world sheet theory. In particular, we can formulate conditions under which the string-derived differential equations reduce to the recursive structure of a generalised holomorphphi anomaly equation. Equations of this type were postulated in [30,31] as the definition of the refined topological string. Finally, we also study the differential equations in the dual setup of heterotic string theory on K3 × T 2. On the heterotic side, the FT¯ gun start receiving contributions at the one-loop level and therefore constitute the ideal testing-ground for the ideas developed in type II, particularly for certain decompactification limits. We find that in the large volume limit of T 2, they satisfy recursive differential equations which precisely match with the weak coupling version of our d

freetail equations in type II, hence providing a non-trivial check of our approach. On the other hand, we use the heterotic setup to study boundary conditions to the differential equations deeloped in this work. Indeed, in [30,31], the field theory limit was used as a boundary condition to solve for the couplings Fagan. In the present case, while the equations in type II are essentially covariant with respect to the choice of vector multiple insertion in Fagan, only the specific choice of the T¯-vector for FT¯ gun was found in [18] to reproduce the gauge theory partition function. Here, we show that also other choices of vector multiple insertions lead to the same boundary Conditons when expanded around an appropriate point of enhanced gauge symmetry in the heterotic moduli space. The paper is organised as follows. In Section 2, we prepare the ground by discussing the effecttie action couplings Fagan and extract several relations among them implied by supersymmetry. In Section 3, we derive equations in type II string theory compactified on a Chalabi–Yau threefold. We derive all necessary amplitudes at higher genus and identify string theoretic corrections to the supergravity equations as boundary terms of the world sheet moduli space. In Section 4, we discuss simplifications of the differential equations which we propose to be the effect of U (1) isometries of the target space Chalabi–Yau threefold. In particular, we point out that, under certain conditions, a recursive structure emerges in the equations, both at the supergravity and at the full string level in type II. In Section 5, we consider the dual heterotic theory on K3 × T 2. We first perform a check of the results obtained in type II from the heterotic dual computation and then provide boundary conditions to the differential equations by reproducing the Nekrasov partition function for different vector multiple insertions in Fagan. Finally, Section 6 contains a summary of our results and our conclusions. Several technical results are compiled in three appendices.

String effective couplings

The central object of this paper is a particular class of higher-derivative effective couplings in N = 2 supersymmetric string compactifications to four dimensions, which were considered in [23,14] (see also [2,8,11–13]) in the form of generalised F-terms. In this section, we demonstrata that supersymmetry requires a number of consistency relations among different component couplings

Super space description:

We begin by reviewing the general class of string effective component couplings [14] of the form

$$\int d^4x\, F_{g,n}^{I_1\ldots I_{2n}}(\varphi,\bar{\varphi})\, R_{(-)}^2 \left(F_{(-)}^G\right)^{2g-2} \left(F_{(+)}^{I_1}\cdot F_{(+)}^{I_2}\right)\ldots\left(F_{(+)}^{I_{2n-1}}\cdot F_{(+)}^{I_{2n}}\right), \qquad (2.1)$$

where R (−) is the (self-dual) Riemann tensor, F G (−) the (self-dual) graviphoton field-strength tensor and FI (+) the (anti-self-dual) field strength tensor of a physical vector multiple gauge field AI μ, which we label by the index I, with I = 1, NV, and μ is a space–time Lorentz index. In general, the coupling function F I1...I2n gun depends covariant on the vector multiple moduli in a non-holomorphic fashion. Only the case n = 0 is special, for which (2.1) reduces to a series of holomorphic couplings [2] of the vector multiple moduli. The supersymmetric version of the component terms (2.1) can be described in standard usper space R4|8 parametrised by the coordinates (Xu, θ a α, θ¯α˙ a).

To this end, we introduce the N = 2 supergravity multiple

$$W_{\mu\nu}^{ab} = \epsilon^{ab}\left(F_{(-)}^{G}\right)_{\mu\nu} + \ldots + (\theta^a\sigma^{\rho\tau}\theta^b)R_{(-)\mu\nu\rho\tau} \qquad (2.2)$$

as well as the chiral and anti-chiral vector multiplets

$$X^I = \varphi^I + \theta_a^\alpha \lambda_\alpha^{Ia} + \epsilon_{ab}(\theta^a\sigma^{\mu\nu}\theta^b)\, F_{(-)\mu\nu}^I + \ldots, \qquad (2.3)$$

$$\bar{X}^I = \bar{\varphi}^I + \bar{\theta}_{\dot{\alpha}}^a \bar{\lambda}_a^{I\dot{\alpha}} + \epsilon^{ab}(\bar{\theta}_a\bar{\sigma}^{\mu\nu}\bar{\theta}_b)\, F_{(+)\mu\nu}^I + \ldots. \qquad (2.4)$$

In addition, we define the descendent fields

$$\bar{K}_{\mu\nu}^I = \left(\epsilon_{ab}\bar{D}^a\bar{\sigma}_{\mu\nu}\bar{D}^b\right)\bar{X}^I = F_{(+)\mu\nu}^I + \ldots, \qquad (2.5)$$

where $\bar{D}_{\dot{\alpha}}^a$ are the (anti-)chiral spinor derivatives. On-shell, these descendants are chiral objects in the sense that

$$\bar{D}_{\dot{\alpha}}^i \bar{K}_{\mu\nu}^I = 0. \qquad (2.6)$$

We can use these superfields to write a superspace version of the component couplings (2.1):

$$\int d^4x\, d^4\theta\, (\bar{D}^a\bar{\sigma}_{\mu\nu}\bar{D}_a)^2 \left[\mathbb{F}_{g,n}^{I_1\ldots I_{2n-2}}(X,\bar{X})\,(W_{\mu\nu}^{ab}W_{ab}^{\mu\nu})^g\,(\bar{K}^{I_1}\cdot\bar{K}^{I_2})\ldots(\bar{K}^{I_{2n-3}}\cdot\bar{K}^{I_{2n-2}})\right]. \qquad (2.7)$$

The non-holomorphic coefficient functions FI1...I2n−2-gun (XI, X¯ I) in (2.7) are generic symmetric tensors transforming in some (reducible) representation of the T-duality group. Upon expansion in the Grassmann variables, they can be related to coefficient couplings, which in turn are related to scattering amplitudes that we study in Section 3 in type II string theory.

Differential equations in type II:

In this section, we consider realisations of the couplings (2.13) discussed above as genus-g string amplitudes in type II string theory on a generic Chalabi–Yau threefold X and derive genreallocations of the relations (2.16) and (2.17) in the far

Gauge field amplitudes:

We begin by providing an expression for the Fagan, ¯ I and then proceed to consider the differenttail equations they satisfy. The key ingredient to deriving the couplings Fagan is the vertex operator of the (anti-self-dual) vector multiple gauge field strength tensor F

. In the −1 2 ghost pictures, it takes the form

$$V_{\star}^{(-1/2)}(z,\bar{z}) = \eta^{\mu} p^{\nu} e^{-\frac{1}{2}(\hat{\varphi}+\hat{\tilde{\varphi}})} (S\bar{\sigma}_{\mu\nu}\tilde{S})\, \Sigma_{\star}(z,\bar{z})\, e^{ip\cdot Z}, \tag{3.1}$$

where z is the insertion point of the vertex on the world sheet, ϕˆ (˜ ϕˆ) are the left-(right-)moving super ghost fields and S (S˜) are the left-(right-)moving space–time spin fields. Furthermore, and are the polarisation and space–time momentum respectively (which satisfy η · p = 0) and Ze are the space–time coordinates. The nature of the vector multiple is determined by the internal field Concretely, upon ozonising the U (1) Kac–Moody currents J and J˜ in terms of H and H˜ respectively, we can write

$$\Sigma_{\star}(z,\bar{z}) = \lim_{w\to z} |w-z|\, e^{\frac{\sqrt{3}i}{2}\left(H(w)\mp\tilde{H}(\bar{w})\right)} \bar{\phi}_{\star}(w,\bar{w}), \tag{3.2}$$

where φ¯ is an (anti-chiral, chiral) primary ((anti-chiral, anti-chiral) primary) state of the type IIA (type IIB) world sheet theory. Assuming that the vector multiple gauge field have no contact terms among themselves,3 which would require the subtraction of 1PI reducible diagrams, the g-loop amplitude can be written [14] in the form of a twisted world sheet correlator integrated over the moduli space Megan of a genus g Riemann surface gun with n punctures (located at positions u)

$$F_{g,n} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{k=1}^{3g-3+n} |\mu_k \cdot G^-|^2 \left(\prod_{m=1}^{n} \int_{\Sigma_{g,n}} \bar{\phi}_{\star}(z_m) \right) \left(\prod_{\ell=1}^{n} \hat{\phi}_{\star}(u_\ell) \right) \right\rangle_{\text{twist}}. \tag{3.3}$$

Our notation allows us to treat type IIA and type IIB string theory simultaneously. Indeed, for the measure, we use the shorthand

$$|\mu \cdot G^-|^2 := \begin{cases} G^-(\mu)\tilde{G}^+(\bar{\mu}) \ldots \text{type IIA}, \\ G^-(\mu)\tilde{G}^-(\bar{\mu}) \ldots \text{type IIB}, \end{cases} \tag{3.4}$$

as a U (1) Kac–Moody current J (J˜). More details, including the algebra relations between all operators, are compiled in Appendix A. Furthermore, the insertions φ

Operator	Charge IIA	Twisted dim IIA	Charge IIB	Twisted dim IIB
G^+	(1,0)	(1,0)	(1,0)	(1,0)
$\tilde{G}^+$	(0,1)	(0,2)	(0,1)	(0,1)
G^-	(−1,0)	(2,0)	(−1,0)	(2,0)
$\tilde{G}^-$	(0,−1)	(0,1)	(0,−1)	(0,2)
$\bar{\phi}_\star$	(−1,1)	(1,1)	(−1,−1)	(1,1)
$\hat{\phi}_\star$	(2,−2)	(0,0)	(2,2)	(0,0)
ρ	(3,0)	(0,0)	(3,0)	(0,0)
$\tilde{\rho}$	(0,−3)	(0,0)	(0,3)	(0,0)

Notice in particular that the total charges of all insertions in (3.3) add up to (−3g + 3, ±3g ∓ 3) in the type IIA (type IIB) theory, as is appropriate for a g-loop correlator. The amplitudes defined in (2.13) can be computed in a similar manner as the Fagen. The only difference is that one of the Am gauge fields is replaced by a different vector multiple a ı¯. At the level of the vertex operators, we simply replace the internal state φ¯ in (3.1) and (3.2) by another (anti-chiral, chiral) primary ((anti-chiral, anti-chiral) primary) state in¯ of the type IIA (type IIB) world sheet theory. Assuming that the gauge fields a and a ı¯ have no contact terms among themselves, we can immediately write the following expression for the amplitude in terms of a twisted world sheet correlation function

$$F_{g,n,\bar{\imath}} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{k=1}^{3g-3+n} |\mu_k \cdot G^-|^2 \left(\prod_{m=1}^{n-1} \int_{\Sigma_{g,n}} \bar{\phi}_{\star}(z_m) \right) \left(\int_{\Sigma_{g,n}} \bar{\phi}_{\bar{\imath}}(z_0) \right) \left(\prod_{\ell=1}^{n} \hat{\phi}_{\star}(u_\ell) \right) \right\rangle_{\text{twist}}. \tag{3.6}$$

Since the charges and (twisted) dimensions of in¯ are identical to φ¯, the total charges of all insertions again add up to (−3g + 3, ±3g ∓ 3) respectively. 3.1.2. Gaugin amplitudes Besides the amplitudes (3.3) and (3.6) presented above, the differential equations (2.16) and (2.17) predicted by supergravity also involve gun defined in (2.13). The latter has two insections of gauging λ¯ α˙ a, whose vertex operators can be obtained from (3.1) by the action o

$$\Psi_{(\star\star|\bar{\imath})}^{g,n} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{k=1}^{3g-3+n} |\mu_k \cdot G^-|^2 \left(\prod_{m=1}^{n-2} \int_{\Sigma_{g,n}} \bar{\phi}_{\star} \right) \left(\int_{\Sigma_{g,n}} \oint G^+ \bar{\phi}_{\star} \right) \left(\int_{\Sigma_{g,n}} \oint \tilde{G}^{\mp} \bar{\phi}_{\star} \right) \times \left(\int_{\Sigma_{g,n}} \bar{\phi}_{\bar{\imath}} \right) \left(\prod_{\ell=1}^{n} \hat{\phi}_{\star}(u_\ell) \right) \right\rangle_{\text{twist}}. \tag{3.7}$$

Differential equations:

Having written the relevant couplings in the form of correlation functions of the twisted type II world sheet theory, we can now derive the stringy analogue of equations (2.16) and (2.17). In the framework of the twisted world sheet correlation functions, an anti-holomorphic moduli derivative D¯ ¯I corresponds to an insertion of the following operator

$$\begin{aligned}
&\text{type IIA:} \quad -\oint G^{+}\oint \tilde{G}^{-}\bar{\phi}_{\bar{l}}, \quad \text{charge}(\bar{\phi}_{\bar{l}})=(-1,+1), \quad \dim(\bar{\phi}_{\bar{l}})=(1,1),\\
&\text{type IIB:} \quad -\oint G^{+}\oint \tilde{G}^{+}\bar{\phi}_{\bar{l}}, \quad \text{charge}(\bar{\phi}_{\bar{l}})=(-1,-1), \quad \dim(\bar{\phi}_{\bar{l}})=(1,1).
\end{aligned} \tag{3.8}$$

These types of deformations of the (twisted) world sheet theory are explained in Appendix A.3, where also our notation for the chiral ring is presented. Notice that I is a (chiral, anti-chiral) primary state in type IIA and a (chiral, chiral) primary state in the type IIB theory. Thus, the left-hand side of equation (2.17) takes the following form (for convenience, we adopt a streamlined shorthand notation):

$$\bar{D}_{\bar{l}}F_{g,n}=-\int_{\mathcal{M}_{g,n}}\left\langle\prod_{k=1}^{3g-3+n}|\mu_k\cdot G^{-}|^2\left(\int\bar{\phi}_{\star}\right)^n\left(\hat{\phi}_{\star}\right)^n\left(\int\oint G^{+}\oint\tilde{G}^{\mp}\bar{\phi}_{\bar{l}}\right)\right\rangle_{\text{twist}}, \tag{3.9}$$

where we are treating the type IIA and type IIB theory simultaneously. Since in the twisted theory (G+, G˜ ∓) have dimensions one, we can deform the corresponding contour integrals to encircle different operators in the correlator. We have

$$\oint G^{+}\hat{\phi}_{\star}=\oint\tilde{G}^{-}\hat{\phi}_{\star}=0 \qquad \text{in type IIA}, \tag{3.10}$$

$$\oint G^{+}\hat{\phi}_{\star}=\oint\tilde{G}^{+}\hat{\phi}_{\star}=0 \qquad \text{in type IIB}, \tag{3.11}$$

due to fact that has charge (+2, ∓2). However, there is a non-trivial residue when G+ or G˜ ∓ encircles φ¯ or one of the operators of the integral measure. Therefore, we find the following contributions I

$$\begin{aligned}
\bar{D}_{\star}F_{g,n}=n\int_{\mathcal{M}_{g,n}}&\left\langle\prod_{k=1}^{3g-3+n}|\mu_k\cdot G^{-}|^2\left(\int\bar{\phi}_{\star}\right)^{n-1}\left(\hat{\phi}_{\star}\right)^n\left(\int\oint G^{+}\bar{\phi}_{\star}\right)\right.\\
&\times\left.\left(\int\oint\tilde{G}^{\mp}\bar{\phi}_{\star}\right)\right\rangle_{\text{twist}}\\
+\int_{\mathcal{M}_{g,n}}&\left\langle\sum_{r=1}^{3g-3+n}\prod_{k\neq r}|\mu_k\cdot G^{-}|^2(\mu_r\cdot T)(\bar{\mu}_r\cdot\tilde{G}^{\pm})\left(\int\bar{\phi}_{\star}\right)^n\left(\hat{\phi}_{\star}\right)^n\right.\\
&\times\left.\left(\int\oint\tilde{G}^{\mp}\bar{\phi}_{\star}\right)\right\rangle_{\text{twist}}.
\end{aligned} \tag{3.12}$$

The first line in this relation corresponds to the amplitude gun (The second line has an insertion of the (left-moving) energy momentum tensor sewed with one of the Belt rami differentials. As we discuss in the next section, such a term can be written as a total derivative [4] in the moduli space Megan and therefore corresponds to a boundary contribution Cody

In a similar fashion as in (3.12), we can write

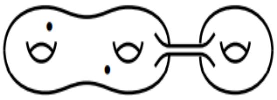
$$\begin{aligned}
\bar{\mathcal{D}}_{\bar{\imath}}F_{g,n}=-n\int_{\mathcal{M}_{g,n}}&\left\langle\prod_{k=1}^{3g-3+n}|\mu_k\cdot G^{-}|^2\left(\int\bar{\phi}_{\star}\right)^{n-1}\left(\int\oint G^{+}\oint\tilde{G}^{\mp}\bar{\phi}_{\star}\right)\left(\hat{\phi}_{\star}\right)^n\right.\\
&\times\left.\int\bar{\phi}_{\bar{\imath}}\right\rangle_{\text{twist}}-n(n-1)\int_{\mathcal{M}_{g,n}}\left\langle\prod_{k=1}^{3g-3+n}|\mu_k\cdot G^{-}|^2\left(\int\bar{\phi}_{\star}\right)^{n-2}\right.\\
&\times\left.\left(\int\oint G^{+}\bar{\phi}_{\star}\right)\left(\int\oint\tilde{G}^{\mp}\bar{\phi}_{\star}\right)\left(\hat{\phi}_{\star}\right)^n\int\phi_{\bar{\imath}}\right\rangle_{\text{twist}}+\mathcal{C}_{\bar{\imath}}^{\text{bdy}},
\end{aligned} \tag{3.14}$$

where for the boundary contribution we write $\mathcal{C}_{\bar{\imath}}^{\text{bdy}}=\mathcal{C}_{\bar{\imath}}^{\text{bdy},1}+\mathcal{C}_{\bar{\imath}}^{\text{bdy},2}$, with

$$\begin{aligned}
\mathcal{C}_{\bar{\imath}}^{\text{bdy},1}&=\int_{\mathcal{M}_{g,n}}\left\langle\sum_{r=1}^{3g-3+n}\prod_{k\neq r}|\mu_k\cdot G^{-}|^2(\mu_r\cdot T)(\bar{\mu}_r\cdot\tilde{T})\left(\int\bar{\phi}_{\star}\right)^n\left(\hat{\phi}_{\star}\right)^n\int\bar{\phi}_{\bar{\imath}}\right\rangle_{\text{twist}},\\
\mathcal{C}_{\bar{\imath}}^{\text{bdy},2}&=-n\int_{\mathcal{M}_{g,n}}\left\langle\sum_{r=1}^{3g-3+n}\prod_{k\neq r}|\mu_k\cdot G^{-}|^2(\mu_r\cdot T)(\bar{\mu}_r\cdot\tilde{G}^{\pm})\left(\int\bar{\phi}_{\star}\right)^{n-1}\left(\hat{\phi}_{\star}\right)^n\right.\\
&\qquad\times\left.\left(\int\oint\tilde{G}^{\mp}\bar{\phi}_{\star}\right)\int\phi_{\bar{\imath}}\right\rangle_{\text{twist}}\\
&\quad-n\int_{\mathcal{M}_{g,n}}\left\langle\sum_{r=1}^{3g-3+n}\prod_{k\neq r}|\mu_k\cdot G^{-}|^2(\mu_r\cdot G^{-})(\bar{\mu}_r\cdot\tilde{T})\left(\int\bar{\phi}_{\star}\right)^n\left(\hat{\phi}_{\star}\right)^n\right.\\
&\qquad\times\left.\left(\int\oint G^{+}\bar{\phi}_{\star}\right)\int\phi_{\bar{\imath}}\right\rangle_{\text{twist}}.
\end{aligned} \tag{3.15}$$

Boundary contributions: The terms Cody and Cody ı¯ introduced above contain energy momentum tensors sewed with the Belt rami differentials. The latter can be re-written as partial derivatives with respect to the local coordinates of Megan and are thus total derivatives. However, Cody ¯ I are not zero, as one might naïvely conclude, due to the contributions from boundaries of Megan. Geometrically, these boundaries correspond to degenerations of gun of which there are three different types

• pinching of a dividing geodesic:



• pinching of a handle:

• collision of two punctures:

The first two contributions can be treated in the same manner as in [4] and are discussed in detail in Appendix B. The collision of two punctures is more involved and is proportional to the curvature on the world sheet, and is not discussed explicitly. However, we remark that its contribution is proportional to Cand turns out to play no role in our later considerations. Summarising the boundary terms, we find

$$C_\star^{\text{bdy}} = \frac{1}{2}C_\star{}^{JK}\left(\sum_{g',n'}{}' \mathcal{D}_J F_{g',n'}\mathcal{D}_K F_{g-g',n-n'} + \mathcal{D}_J\mathcal{D}_K F_{g-1,n}\right)$$

$$+\,(\text{curvature contributions}),$$

$$C_{\bar\imath}^{\text{bdy}} = -\frac{1}{2}n\,C_\star{}^{JK}\left(\sum_{g',n'}{}' \mathcal{D}_J F_{g',n',\bar\imath}\mathcal{D}_K F_{g-g',n-n'} + \mathcal{D}_J\mathcal{D}_K F_{g-1,n,\bar\imath}\right)$$

$$+\frac{1}{2}C_{\bar\imath}{}^{JK}\left(\sum_{g',n'}{}' \mathcal{D}_J F_{g',n'}\mathcal{D}_K F_{g-g',n-n'} + \mathcal{D}_J\mathcal{D}_K F_{g-1,n}\right)$$

where K is the Kähler potential. In addition, and throughout the manuscript, the notation ——————————— means that we exclude the terms (0, 0), (0, 1), (g, n − 1) and (g, n) from the summation range. Combined with (3.16), this gives rise to the following equations which are valid at a generic point in the full string moduli space up to curvature contributions

$$\bar{\mathcal{D}}_\star F_{g,n} = n\Psi^{g,n}_{(\star\star\star)} + \frac{1}{2}C_\star{}^{JK}\left(\sum_{g',n'}{}' \mathcal{D}_J F_{g',n'}\mathcal{D}_K F_{g-g',n-n'} + \mathcal{D}_J\mathcal{D}_K F_{g-1,n}\right), \tag{3.18}$$

$$\begin{aligned}\bar{\mathcal{D}}_{\bar\imath} F_{g,n} &= n\bar{\mathcal{D}}_\star F_{g,n,\bar\imath} - n(n-1)\Psi^{g,n}_{(\star\star|\bar\imath)}\\ &\quad -\frac{n}{2}C_\star{}^{JK}\left(\sum_{g',n'}{}' \mathcal{D}_J F_{g',n',\bar\imath}\mathcal{D}_K F_{g-g',n-n'} + \mathcal{D}_J\mathcal{D}_K F_{g-1,n,\bar\imath}\right)\\ &\quad +\frac{1}{2}C_{\bar\imath}{}^{JK}\left(\sum_{g',n'}{}' \mathcal{D}_J F_{g',n'}\mathcal{D}_K F_{g-g',n-n'} + \mathcal{D}_J\mathcal{D}_K F_{g-1,n}\right).\end{aligned} \tag{3.19}$$

Notice, as a consistency check, that (3.19) reduces to (3.18) once ı¯ is taken to be from the point of view of supergravity, apart from the boundary contribution, the equations (3.18) and (3.19) agree with the predictions from supersymmetry. In general, 'anomalous' contributions like Cody or Cody ı¯ are beyond the simple on-shell analysis performed in Section 2.2, as was pointed out in [4–13]. On the other hand, from the point of view of topological string theory, the derivatives D¯ ¯I lead to the insertion of the operators (3.8) into the correlator Fagen, which is outside the BRSTchorology. Therefore, whenever the right-hand sides of (3.18) and (3.19) vanish (up to the anomalous boundary contributions), the Fagen may be interpreted as topological objects. The presence of the Megan bulk terms in D indicates that the couplings Fagen genreicily receive contributions from non-physical states in the topologically twisted theory. In the full string theory, this corresponds to the observation that the Fagen are not BPS-saturated quantaties, but also receive contributions from non-BPS states. This can be seen from the formulation of the couplings in (2.7): they are not (BPS-saturated) F-terms, but are rather D-terms, with the d4θ (D¯ · D) ¯ 2 essentially acting as an integration over the full R4|8 standard super space. However, we note that the situation changes for n = 0. Indeed, in this case, equations (3.18) and (3.19) reduce to the holomorphic anomaly equation [4] for the topological amplitudes Fig, discussed in [2]. The equation then encodes the stronger property that the couplings Fig are holomorphic functions of the vector multiples moduli [2].

Non-compact limit:

Although we just explained that the correlation functions (3.3) (for n = 0) are generically nontopological, we argued in [18] that the string couplings Fagen, with identified with the vector 264 I. Antoniadis et al. / Nuclear Physics B 901 (2015) 252–281 super partner of the Kähler modulus of the dual heterotic K3×T 2 theory, possess numerous propretires one would expect from a world sheet realisation of the genus g free energy of the refined topological string. Most importantly, we showed that when expanded around a particular point in the string moduli space, the Fagen reproduce in the point-particle limit the (perturbative part) of Nekrasov's partition function in the general - background. This result was extended beyond the perturbative level in [24] and is conceptually a very strong check of our proposal. Given this evidence, it is an interesting and important question to study whether the Fagen can be rendered topological in some appropriate limit in the physical moduli space in which the non-physical states (from the point of view of the twisted theory) decouple in the world sheet description.

This would lead to a vanishing of the bulk contributions in the right-hand side of (3.18) and (3.19). In the framework of supergravity, this corresponds to rendering the effective couplings Fagen in (2.13) holomorphic, such that the respective couplings (2.1) are BPS-saturated. The necessity of taking such a limit seems rather natural from the point of view of the gauge theory. Indeed, formulating the -background in four-dimensional space–time requires the pressonce of a U (1) isometry in the internal manifold. Such isometries are generically not present in compact Chalabi–Yau threefold but may arise in non-compact ones. Therefore, we expect that in an appropriate non-compact limit, the differential equations (3.18) and (3.19) are simplified due to the presence of additional Ward identities ascribed to the emergent U (1) isometry, such that the Fagen become topological objects. In the following, we analyse necessary conditions for this to occur from the point of view of supergravity and of type II string theory

Supergravity conditions:

The conditions (2.14) and (2.15) derived in supergravity are solely a consequence of usper symmetry and the structure of the coupling (2.7). In particular, they do not simply encode properties of single tensor components FI1...I2n−2 gun as a function of the vector multiples. Rather, once these relations are translated into the language of scattering amplitudes (2.16) and (2.17), they relate several different objects, instead of just a single type of them and thus give rise to the bulk terms. In the following, we derive a set of consistent conditions that can be imposed on the component functions Fagen, if¯ gun, Fiji¯ gun etc. directly, such that the resulting equations only involve a single class of objects. More specifically, at the level of the amplitudes we impose that both sides in (2.16) vanish separately

Heterotic realisation:

The results of the previous sections lend further support to our proposal that the Fagen studied in [18] can furnish a world sheet description of the refined topological string for specific choices of the internal manifold or in suitable decompactification limits. The crucial property for these couplings is that in the point particle limit, the Fagen reproduce Nekrasov's gauge theory partiton function on the full - background, with both deformation parameters being non-trivial. In [18,24], working in the dual heterotic theory on K3 × T 2, we showed that the Fagen involving insertions of field-strengths in the vector multiples of the T 2 Kähler modulus correctly reproduce the perturbative and non-perturbative parts of the Nekrasov partition function when expanded around a particular point in the string moduli space. We denote these couplings by FT¯ gun in the remainder of the section. An important check of the approach described in the previous sections concerns the differenttail equations satisfied by the realisation of the couplings Fagen in the dual heterotic framework on K3 × T 2, since their explicit expression is known by a direct one-loop computation at the full string level [18]. We show in the following that, in the large T 2 volume limit, the equations satisfied by FT¯ gun precisely match with the weak coupling limit of (4.8)

Interpretation and conclusions:

In this paper, we have discussed the class of super space couplings (2.7) in the N = 2 supergravity action. We have analysed consistency conditions between its various component terms that are imposed by supersymmetry. These do not simply constrain the moduli dependence of a single component coupling (e.g., holomorphicity as in the case of n = 0, see [2]), but rather relate different component terms with one another. These relations were formulated as first order differential equations, e.g. (2.16) and (2.17). Based on the evidence in support of our proposal [18] for the Fagen as candidates for the refiremen of the topological string, following [14], we derived all couplings (2.13) as higher loop scattering amplitudes in the framework of type II string theory on a (compact) Chalabi–Yau manfold. These string effective couplings were shown to satisfy (2.16) and (2.17) up to additional terms which arose as boundary contributions of the moduli space of the genus g world sheet with n punctures. The latter play a similar role as the holomorphic anomaly found in [2] in the case of n = 0. The resulting equations (3.18) and (3.19) are solely a consequence of the N = (2, 2) world sheet supersymmetry and hold at a generic point in the string moduli space. Provided cartrain well-defined conditions are met, these equations reduce to a form involving only one type of component couplings and exhibit a recursive structure in both g and n. The resulting equalton (4.8) is structurally similar to the generalised holomorphic anomaly equation proposed in [30,31] as a definition for the free energy of the refined topological string on local/non-compact Chalabi–Yau manifolds. These results support our proposal [18,24] for the couplings Fagen as a world sheet definition of the refined topological string. The present work further analyses the necessary conditions for the validity of our proposal. At a generic point in the moduli space of a (compact) Chalabi–Yau manifold, the couplings Fagen are not BPS-saturated and their (twisted) world sheet representation (3.3) is not topological. This manifests itself in the fact that the Fagen are related to different classes of couplings. We expect that the U (1) isometry, recovered at certain regions in the boundary of moduli space, is responsible for a

simplification of these equations (see e.g. (4.8)) that is appropriate for a topological object. We have provided the well-posed necessary and sufficient conditions (4.9) (formulated in terms of physical quantities only) for this modification to happen. Furthermore, by analysing the explicit form of the Fagen in the dual heterotic theory on K3 × T 2, we obtained perfect agreement with the weak coupling limit of (4.8). An interesting open question concerns the study of explicit examples of Chalabi–Yau geometries and the analysis of the geometric implications of the consistency conditions derived in this work. As was also noted in [30,31], the differential equations are not sufficient to define the partition function of the free energy of the topological string since it must be supplemented by suitable boundary conditions. One such condition is the point particle limit in which the topological free energy, when expanded around a point of enhanced gauge symmetry, should reproduce the partuition function for N = 2 supersymmetric gauge theories in a general -background. In the case of the string couplings Fagen, this limit was analysed perturbatively and non-perturbatively in [18, 24] for a being identified with the vector super partner of the heterotic T¯-modulus of T 2, and indeed the full gauge theory partition function was reproduced. In this work we have extended this analysis and found that all couplings Fagen with φ∈ O (2,10) O (2) ×O (10) reproduce perturbatively I

Nekrasov's partition function, when expanded around an appropriate point of enhanced gauge symmetry in the string moduli space. In summary, the findings of this paper further corroborate our proposal that the string scattiring amplitudes Fagen can provide a world sheet description of the refined topological string. Indeed, we have elucidated the conditions under which such an identification is possible. We have also shown that our proposal is compatible with other approaches towards the refined topologycall string. In particular, starting only from physical quantities (i.e., string scattering amplitudes), we have proposed a way of finding a generalised holomorphic anomaly equation, which e.g., in [30,31] was postulated as the definition of the refined topological string.

Acknowledgements

We thank D. Orlando, S. Refer, S. Saakashvili, E. Sketched and T. Taylor for very interestIng discussions. S.H. would especially like to thank A. Iqbal, H. Jockers, A. Klemm and Soo-Jong Rey for very inspiring and useful conversations and discussions. I.F. would like to thank the ICTP Trieste and A.Z.A. wishes to thank the CERN Theory Division for their warm hospitality during several stages of this work. S.H. would like to thank the ICTP Trieste, the Asia Pacific Centre for Theoretical Physics and Seoul National University for warm hospitality and for creating a stimulating research environment while part of this work was done. The work of S.H. is partly supported by the BQR Acoel EC 2015.

Appendix A. World-sheet super conformal field theory

The two-dimensional N = 2 super conformal algebra of central charge c is spanned by the energy momentum tensor T, two supercurrents G± and a U (1) Kac–Moody current J. The conformal dimensions and the charges of all operators under J are summarised in the following table

Operator	Conf. weight	$U(1)$
T	2	0
$G^{\pm}$	3/2	±1
J	1	0

The algebra is realised through the OPE relations among the different operators, which are

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w},$$

$$T(z)G^{\pm}(w) = \frac{3G^{\pm}(w)}{2(z-w)^2} + \frac{\partial_w G^{\pm}(w)}{z-w},$$

$$T(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w}, \qquad J(z)G^{\pm}(w) = \pm\frac{G^{\pm}(w)}{z-w},$$

$$J(z)J(w) = \frac{c}{3(z-w)^2}, \qquad G^{+}(z)G^{+}(w) = G^{-}(z)G^{-}(w) = 0,$$

$$G^{+}(z)G^{-}(w) = \frac{2c}{3(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w)+\partial_w J(w)}{z-w}. \tag{A.1}$$

Topological twist:

In this work, we study correlators in a topologically twisted version of the world sheet theory discussed above. There are two independent ways to redefine the energy–momentum tensor, which are known as the A- and the B-twist:

$$\text{A-twist:} \quad T \to T - \frac{1}{2}\partial J, \qquad \tilde{T} \to \tilde{T} + \frac{1}{2}\bar{\partial}\tilde{J}, \tag{A.2}$$

$$\text{B-twist:} \quad T \to T - \frac{1}{2}\partial J, \qquad \tilde{T} \to \tilde{T} - \frac{1}{2}\bar{\partial}\tilde{J}. \tag{A.3}$$

These twists have the effect of shifting the dimensions of all operators by (half of) their charge as shown in the table below

Operator	A-twisted dimension	B-twisted dimension
T	(2, 0)	(2, 0)
$\tilde{T}$	(0, 2)	(0, 2)
G^{+}	(1, 0)	(1, 0)
$\tilde{G}^{+}$	(0, 2)	(0, 1)
G^{-}	(2, 0)	(2, 0)
$\tilde{G}^{-}$	(0, 1)	(0, 2)
J	(1, 0)	(1, 0)
$\tilde{J}$	(0, 1)	(0, 1)

With these dimensions, we can identify the operators (G+, G˜ −) with the left- and right-moving BRST operators in the A-twisted theory, and (G+, G˜ +) with the left- and right-moving BRST operators in the B-twisted theory. Physical states of the A- and B-type topological theory are defined to lie in the chorology of the corresponding BRST operators. Similarly, the operators (G−, G˜ +) in the A-twisted model and (G−, G˜ −) in the B-twisted model have the right dimesons to be identified with the anti-ghost operators. Indeed, they have the right dimensions to be sewed with the Beltrami-differentials of a Riemann surface, thus providing an integral measure for the twisted correlators as defined in (3.3).

Appendix C. Lattice momenta

In this appendix, we discuss our conventions for the self-dual lattices which are at the heart of heterotic torus compactifications. The basic moduli in the case of T 2 are the two-dimensional metric gab, the B-field BAB and Wilson-line moduli WA A. The indices A, B = 1, 2 denote the directions on the torus, while a = 1, 8. An explicit parametrisation is given by

$$g_{AB} = \frac{T_2 - \frac{W_2^{\mu}W_2^{\mu}}{2U_2}}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & U_1^2+U_2^2 \end{pmatrix} \quad \text{and} \quad B_{AB} = \left(T_1 - \frac{W_1^{\mu}W_2^{\mu}}{2U_2}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{C.1}$$

where we have used the physical moduli

$$T = T_1 + iT_2, \qquad U = U_1 + iU_2, \qquad W^a = V_2^a - UV_1^a. \tag{C.2}$$

Using these objects, we can define the lattice momenta of the $\Gamma^{2,10}$ self-dual lattice as

$$P_L^A = m^A + V_a^A b^a + \tfrac{1}{2}V_a^A V_a^B n_B + B^{AB}n_B + g^{AB}n_B, \tag{C.3}$$

$$\vec{P}_R = \begin{pmatrix} P_R^a \\ P_R^A \end{pmatrix} = \begin{pmatrix} b^a + V_A^a n^A \\ m^A + V_a^A b^a + \tfrac{1}{2}V_a^A V_a^B n_B + B^{AB}n_B - g^{AB}n_B \end{pmatrix}, \tag{C.4}$$

where n^a, m^a, b^a are integer numbers. These momenta satisfy the relation

$$\tfrac{1}{2}\left(P_L^A g_{AB} P_L^B - P_R^A g_{AB} P_R^B - P_R^a P_R^a\right) = 2(m_1 n_1 + m_2 n_2) - b^a b^a. \tag{C.5}$$

For most of the computations carried out in Section 5, it is useful to work in a complex basis, *i.e.* instead of $(P_L^A;\, P_R^A,\, P_R^a)$ we introduce $(P_L,\, \bar{P}_L;\, P_R,\, \bar{P}_R,\, P_R^a)$. In order to save writing, we also introduce the shorthand notation

$$\xi = \sqrt{(T-\bar{T})(U-\bar{U}) - \frac{1}{2}(W-\bar{W})^2}, \tag{C.6}$$

as well as

$$K_{g,n} \equiv \tau_2^{2g+2n-3}\left(\frac{P_L}{\xi}\right)^{2g-2}\left(\frac{P_R}{\xi}\right)^{2n}\hat{\Gamma}^{(2,10)}, \quad \text{with } \hat{\Gamma}^{2,10} = q^{|P_L|^2}\bar{q}^{|P_R|^2+\frac{1}{2}p^2}. \tag{C.7}$$

After some algebra, one can show the following identities:

$$\partial_{\bar{T}}\hat{\Gamma}_{2,10} = -\frac{4\pi\tau_2(U-\bar{U})}{\xi^2}\bar{P}_L P_R \hat{\Gamma}^{2,10},$$

$$\partial_{\bar{U}}\hat{\Gamma}_{2,10} = -\frac{4\pi\tau_2}{\xi^2(U-\bar{U})} \times\left[\frac{1}{2}(W^a-\bar{W}^a)^2\bar{P}_L P_R + \xi^2\bar{P}_L\bar{P}_R + \xi(W^a-\bar{W}^a)P_R^a\bar{P}_L\right]\hat{\Gamma}^{2,10},$$

$$(\partial_{\bar{W}})^a\hat{\Gamma}_{2,10} = \frac{4\pi\tau_2}{\xi^2}\left[(W^a-\bar{W}^a)\bar{P}_L P_R + \xi P_R^a\bar{P}_L\right]\hat{\Gamma}^{2,10}. \tag{C.8}$$

These allow us to prove that the action of anti-holomorphic derivatives on $K_{g,n}$ is related to that of holomorphic derivatives on $K_{g-1,n}$ up to terms suppressed in the large T_2 limit[9]:

$$\mathcal{D}_{\bar{i}}K_{g,n}\big|_{T_2\to\infty} = -\frac{1}{2\pi i}e^{2K}C_{\bar{i}\bar{j}\bar{s}}G^{j\bar{j}}\partial_\tau\left(\tau_2^2\mathcal{D}_j K_{g-1,n}\right)\big|_{T_2\to\infty}, \tag{C.9}$$

where $\mathcal{D}_{\bar{i}}$ is a suitable Kähler covariant derivative taking into account the weight of $K_{g,n}$.

References

[1] E. Witten, Topological sigma models, Common. Math. Phys. 118 (1988) 411; E. Witten, On the structure of the topological phase of two-dimensional gravity, Null. Phys. B 340 (1990) 281.

[2] I. Antoniadis, E. Gava, K.S. Narain, T.R. Taylor, Topological amplitudes in string theory, Null. Phys. B 413 (1994) 162, arrive: hep-the/9307158.

[3] G. Lopes Cardoso, B. de Wit, T. Mohaupt, Corrections to macroscopic supersymmetric black-hole entropy, Phys. Lett. B 451 (1999) 309, arrive: hep-the/9812082; G. Lopes Cardoso, B. de Wit, T. Mohaupt, Deviations from the area law for supersymmetric black holes, Bortsch. Phys. 48 (2000) 49, arrive: hep-the/9904005; G. Lopes Cardoso, B. dew it, T. Mohaupt, Macroscopic entropy formulae and nonholomorphic corrections for usper symmetric black holes, Null. Phys. B 567 (2000) 87, arrive: hep-the/9906094; G. Lopes Cardoso, B. de Wit, T. Mohaupt, Area law corrections from state counting and supergravity, Class. Quantum Gravity 17 (2000) 1007, arrive: hep-the/9910179; T. Mohaupt, Black hole entropy, special geometry and strings, Bortsch. Phys. 49 (2001) 3, arrive: hep-the/0007195.

[4] M. Bershadsky, S. Cecotti, H. Ogura, C. Vafa, Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes, Common. Math. Phys. 165 (1994) 311, arrive hep-the/9309140.

[5] N. Berkovits, C. Vafa, N = 4 topological strings, Null. Phys. B 433 (1995) 123, arrive: hep-the/9407190.

[6] N. Berkovits, C. Vafa, Type IIB R4 H∗ (4g − 4) conjectures, Null. Phys. B 533 (1998) 181, arrive: hep-the/9803145. [7] H. Ogura, C. Vafa, All loop N = 2 string amplitudes, Null. Phys. B 451 (1995) 121, arrive: hep-the/9505183.

[8] I. Antoniadis, E. Gava, K.S. Narain, T.R. Taylor, Topological amplitudes in heterotic superstring theory, Null. Phys. B 476 (1996) 133, arrive: hep-the/9604077.

[9] I. Antoniadis, K.S. Narain, T.R. Taylor, Open string topological amplitudes and gauging masses, Null. Phys. B 729 (2005) 235, arrive: hep-t0507244.

[10] I. Antoniadis, S. Honegger, K.S. Narain, N = 4 topological amplitudes and string effective action, Null. Phys. B 771 (2007) 40, arrive: hep-the/0610258.

[11] I. Antoniadis, S. Honegger, K.S. Narain, E. Sketched, Harmonicity in N = 4 supersymmetry and its quantum anomaly, Null. Phys. B 794 (2008) 348, arXiv:0708.0482 [hep-the]. 9 In fact, eq. (C.9) is exact except for ı¯ = U¯. 280 I. Antoniadis et al. / Nuclear Physics B 901 (2015) 252–281

[12] I. Antoniadis, S. Honegger, K.S. Narain, E. Sketched, A new class of N = 2 topological amplitudes, Null. Phys. B 823 (2009) 448, arXiv:0905.3629 [hep-the].

[13] I. Antoniadis, S. Honegger, K.S. Narain, E. Sketched, Generalized N = 2 topological amplitudes and holomorphphi anomaly equation, Null. Phys. B 856 (2012) 360, arXiv:1107.0303 [hep-the].

[14] I. Antoniadis, S. Honegger, K.S. Narain, T.R. Taylor, Deformed topological partition function and Nekrasov backgrounds, Null. Phys. B 838 (2010) 253, arXiv:1003.2832 [hep-the].

[15] G.W. Moore, N. Nekrasov, S. Saakashvili, Integrating over Higgs branches, Common. Math. Phys. 209 (2000) 97, arrive: hep-the/9712241.

[16] A. Losev, N. Nekrasov, S.L. Saakashvili, Testing Seiberg–Witten solution, in: Strings, Branes and Dualities, Cargoes, 1997, pp. 359–372, arrive: hep-the/9801061.

[17] N.A. Nekrasov, Seiberg–Witten prepotential from instanton counting, Adv. Theor. Math. Phys. 7 (2004) 831, arrive: hep-the/0206161.

[18] I. Antoniadis, I. Florakis, S. Honegger, K.S. Narain, A. Zein Assi, World sheet realization of the refined topological string, Null. Phys. B 875 (2013) 101, arXiv:1302.6993 [hep-the]. [

19] R. Gopak Umar, C. Vafa, M theory and topological strings. 1, arrive: hep-the/9809187; R. Gopak Umar, C. Vafa, M theory and topological strings. 2, arrive: hep-the/9812127; T.J. Hollowood, A. Iqbal, C. Vafa, Matrix models, geometric engineering and elliptic genera, J. High Energy Phys. 0803 (2008) 069, arrive: hep-the/0310272.